An Introduction to String Theory and D-Brane Dynamics

With Problems and Solutions

2nd Edition

Richard J Szabo

Imperial College Press
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String Theory and
D-Brane Dynamics

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With Problems and Solutions

2nd Edition

Richard J Szabo
Heriot-Watt University, UK
To my Dad, for a good long read
Preface to Second Edition

The second edition of this book has improved in two main ways. Firstly, a few minor typos which plagued the first edition have been corrected. Secondly, and most notably, an extra chapter at the end has been added with detailed solutions to all exercises that appear in the main part of the text. These exercises are meant both to fill in some gaps in the main text, and also to provide the student reader with the rudimentary computational skills required in the field. The inclusion of this extra chapter of solutions should help make this second edition more complete and self-contained. It was decided to keep the overall style of the rest of the book intact, and hence the remaining chapters still serve as a brief, concise and quick introduction into the basic aspects of string theory and D-brane physics.

The author would like to thank Laurent Chaminade and others at Imperial College Press for the encouragement to produce this second edition. This work was supported in part by grant ST/G000514/1 “String Theory Scotland” from the UK Science and Technology Facilities Council.

Richard J. Szabo

*Edinburgh, 2010*
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Preface to First Edition

These notes comprise an expanded version of the string theory lectures given by the author at the 31st and 32nd British Universities Summer Schools on Theoretical Elementary Particle Physics (BUSSTEPP) which were held, respectively, in Manchester, England in 2001 and in Glasgow, Scotland in 2002, and also at the Pacific Institute for the Mathematical Sciences Frontiers of Mathematical Physics Summer School on “Strings, Gravity and Cosmology” which was held in Vancouver, Canada in 2003. The schools were attended mostly by Ph.D. students in theoretical high-energy physics who had just completed their first year of graduate studies. The lectures were thereby appropriately geared for this level. No prior knowledge of string theory was assumed, but a good background in quantum field theory, introductory level particle physics and group theory was expected. An acquaintance with the basic ideas of general relativity was helpful but not absolutely essential. Some familiarity with supersymmetry was also assumed because the supersymmetry lectures proceeded the string theory lectures at the schools, although the full-blown machinery and techniques of supersymmetry were not exploited to any large extent.

The main references for string theory used during the courses were the standard books on the subject [Green, Schwarz and Witten (1987); Polchinski (1998)] and the more recent review article [Johnson (2001)]. The prerequisite supersymmetry lectures can be found in [Figueroa-O’Farrill (2001)]. Further references are cited in the text, but are mostly included for historical reasons and are by no means exhaustive. Complete sets of references may be found in the various cited books and review articles.

The lectures were delivered in the morning and exercises were assigned. These problems are also included in these notes. Many of them are intended to fill in the technical gaps which due to time constraints were not covered in
the lectures. Others are intended to give the student a better grasp of some “stringy” topics. This book has expanded on many aspects of string theory that were addressed during the schools, mainly to make the presentation clearer.

There were six one-hour lectures in total. Since string theory is nowadays such a vast and extensive subject, some focus on the subject material was of course required. The lectures differ perhaps from most introductory approaches since the intent was to provide the student not only with the rudiments of perturbative string theory, but also with an introduction to the more recently discovered non-perturbative degrees of freedom known as “D-branes”, which in the past few years have revolutionized the thinking about string theory and have brought the subject to the forefront of modern theoretical particle physics once again. This means that much of the standard introductory presentation was streamlined in order to allow for an introduction to these more current developments. The hope was that the student will have been provided with enough background to feel comfortable in starting to read current research articles, in addition to being exposed to some of the standard computational techniques in the field.

The basic perturbative material was covered in roughly the first three lectures and comprises chapters 1–4. Lecture 4 (chapter 5) then started to rapidly move towards explaining what D-branes are, and at the same time introducing some more novel stringy physics. Lectures 5 and 6 (chapters 6 and 7) then dealt with D-branes in detail, studied their dynamics, and provided a brief account of the gauge theory/string theory correspondence which has been such an active area of research over the past few years. For completeness, an extra chapter has also been added (chapter 8) which deals with the Ramond–Ramond couplings of D-branes and other novel aspects of D-brane dynamics such as the important “branes within branes” phenomenon. This final chapter is somewhat more advanced and is geared at the reader with some familiarity in differential topology and geometry.

The author is grateful to the participants, tutors and lecturers of the schools for their many questions, corrections, suggestions and criticisms which have all gone into the preparation of these lecture notes. He would especially like to thank M. Alford, C. Davies, J. Forshaw, M. Rozali and G. Semenoff for having organised these excellent schools, and for the encouragement to write up these notes. He would also like to thank J. Figueroa-O’Farrill and F. Lizzi for practical comments on the manuscript.
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Richard J. Szabo

*Edinburgh, 2003*
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Chapter 1

A Brief History of String Theory

To help introduce the topics which follow and their significance in high energy physics, in this chapter we will briefly give a non-technical historical account of the development of string theory to date, focusing on its achievements, failures and prospects. This will also help to motivate the vast interest in string theory within the particle theory community. It will further give an overview of the material which will follow.

In conventional quantum field theory, the fundamental objects are mathematical points in spacetime, modeling the elementary point particles of nature. String theory is a rather radical generalization of quantum field theory whereby the fundamental objects are extended one-dimensional lines or loops (Fig. 1.1). The various elementary particles observed in nature correspond to different vibrational modes of the string. While we cannot see a string (yet) in nature, if we are very far away from it we will be able to see its point-like oscillations, and hence measure the elementary particles that it produces. The main advantage of this description is that while there are many particles, there is only one string. This indicates that strings could serve as a good starting point for a unified field theory of the fundamental interactions.

This is the idea that emerged by the end of the 1960s from several years of intensive studies of dual models of hadron resonances [Veneziano (1968)]. In this setting, string theory attempts to describe the strong nuclear force. The excitement over this formalism arose from the fact that string S-matrix scattering amplitudes agreed with those found in meson scattering experiments at the time. The inclusion of fermions into the model led to the notion of a supersymmetric string, or “superstring” for short [Neveu and Schwarz (1971); Ramond (1971)]. The massive particles sit on “Regge trajectories” in this setting.
However, around 1973 the interest in string theory quickly began to fade, mainly because quantum chromodynamics became recognized as the correct quantum field theory of the strong interactions. In addition, string theories possessed various undesirable features which made them inappropriate for a theory of hadrons. Among these were the large number of extra spacetime dimensions demanded by string theory, and the existence of massless particles other than the spin 1 gluon in the spectrum of string states.

In 1974 the interest in string theory was revived for another reason [Scherk and Schwarz (1974); Yoneya (1974)]. It was found that, among the massless string states, there is a spin 2 particle that interacts like a graviton. In fact, the only consistent interactions of massless spin 2 particles are gravitational interactions. Thus string theory naturally includes general relativity, and it was thereby proposed as a unified theory of the fundamental forces of nature, including gravity, rather than a theory of hadrons. This situation is in marked contrast to that in ordinary quantum field theory, which does not allow gravity to exist because its scattering amplitudes that involve graviton exchanges are severely plagued by non-renormalizable ultraviolet divergences (Fig. 1.2). On the other hand, string theory is a consistent quantum theory, free from ultraviolet divergences, which necessarily requires gravitation for its overall consistency.

With these facts it is possible to estimate the energy or length scale at which strings should be observed in nature. Since string theory is a relativistic quantum theory that includes gravity, it must involve the corresponding three fundamental constants, namely the speed of light $c$, the reduced Planck constant $\hbar$, and the Newtonian gravitational constant $G$. These three constants may combined into a constant with dimensions of length. The characteristic length scale of strings may thereby be estimated.
by the Planck length of quantum gravity:

\[ \ell_P = \left( \frac{\hbar G}{c^3} \right)^{3/2} = 1.6 \times 10^{-33} \text{ cm} \, . \] (1.1)

This is to be compared with the typical size of hadrons, which is of the order of $10^{-13}$ cm. The corresponding energy scale is known as the Planck mass:

\[ m_P = \left( \frac{\hbar c}{G} \right)^{1/2} = 1.2 \times 10^{19} \text{ GeV}/c^2 \, . \] (1.2)

These scales indicate the reasons why strings have not been observed in nature thus far. Present day particle accelerators run at energies $\ll m_Pc^2$ and thus cannot resolve distances as short as the Planck length. At such energies, strings look like point particles, because at very large distance scales compared to the Planck length all one can observe is the string’s center of mass motion, which is point-like. Thus at these present day scales, strings are accurately described by quantum field theory.

For many of the subsequent years superstring theory began showing great promise as a unified quantum theory of all the fundamental forces including gravity. Some of the general features which were discovered are:

- General relativity gets modified at very short distances/high energies (below the Planck scale), but at ordinary distances and energies it is present in string theory in exactly the same form as Einstein’s theory.
• “Standard model type” Yang–Mills gauge theories arise very naturally in string theory. However, the reasons why the gauge group $SU(3) \times SU(2) \times U(1)$ of the standard model should be singled out is not yet fully understood.

• String theory predicts supersymmetry, because its mathematical consistency depends crucially on it. This is a generic feature of string theory that has not yet been discovered experimentally.

This was the situation for some years, and again the interest in string theory within the high energy physics community began to fade. Different versions of superstring theory existed, but none of them resembled very closely the structure of the standard model.

Things took a sharp turn in 1985 with the birth of what is known as the “first superstring revolution”. The dramatic achievement at this time was the realization of how to cancel certain mathematical inconsistencies in quantum string theory. This is known as Green–Schwarz anomaly cancellation [Green and Schwarz (1984)] and its main consequence is that it leaves us with only five consistent superstring theories, each living in ten spacetime dimensions. These five theories are called Type I, Type IIA, Type IIB, $SO(32)$ heterotic, and $E_8 \times E_8$ heterotic. The terminology will be explained later on. For now, we simply note the supersymmetric Yang–Mills gauge groups that arise in these theories. The Type I theories have gauge group $SO(32)$, both Type II theories have $U(1)$, and the heterotic theories have gauge groups as in their names. Of particular phenomenological interest was the $E_8 \times E_8$ heterotic string, because from it one could construct grand unified field theories starting from the exceptional gauge group $E_6$.

The spacetime dimensionality problem is reconciled through the notion of “compactification”. Putting six of the spatial directions on a “small” six-dimensional compact space, smaller than the resolution of the most powerful microscope, makes the 9+1 dimensional spacetime look 3+1 dimensional, as in our observable world. The six dimensional manifolds are restricted by string dynamics to be “Calabi–Yau spaces” [Candelas et al (1985)]. These compactifications have tantalizingly similar features to the standard model. However, no complete quantitative agreement has been found yet between the two theories, such as the masses of the various elementary particles. This reason, and others, once again led to the demise of string theory towards the end of the 1980s. Furthermore, at that stage one only understood how to formulate superstring theories in terms of divergent perturbation series analogous to quantum field theory. Like in quantum chromodynamics, it
is unlikely that a realistic vacuum can be accurately analysed within perturbation theory. Without a good understanding of nonperturbative effects (such as the analogs of QCD instantons), superstring theory cannot give explicit, quantitative predictions for a grand unified model.

This was the state of affairs until around 1995 when the “second superstring revolution” set in. For the first time, it was understood how to go beyond the perturbation expansion of string theory via “dualities” which probe nonperturbative features of string theory [Font et al (1990); Hull and Townsend (1995); Kachru and Vafa (1995); Schwarz (1995); Sen (1994)]. The three major implications of these discoveries were:

- **Dualities relate all five superstring theories in ten dimensions to one another.**

The different theories are just perturbative expansions of a unique underlying theory $\mathcal{U}$ about five different, consistent quantum vacua [Schwarz (1996); Schwarz (1997)]. Thus there is a completely unique theory of nature, whose equation of motion admits many vacua. This is of course a most desirable property of a unified theory.

- **The theory $\mathcal{U}$ also has a solution called “M-Theory” which lives in 11 spacetime dimensions [Duff (1996); Townsend (1995); Witten (1995)].**

The low-energy limit of M-Theory is 11-dimensional supergravity [Cremmer, Julia and Scheck (1978)]. All five superstring theories can be thought of as originating from M-Theory [Duff (1996); Schwarz (1997)] (see Fig. 1.3). The underlying theory $\mathcal{U}$ is depicted in Fig. 1.4.

- **In addition to the fundamental strings, the theory $\mathcal{U}$ admits a variety of extended nonperturbative excitations called “p-branes”, where $p$ is the number of spatial extensions of the objects [Horowitz and Strominger (1991)].**

Especially important in this regard are the “Dirichlet p-branes” [Dai, Leigh and Polchinski (1989); Hořava (1989); Polchinski (1995)], or “D-branes” for short, which are $p$-dimensional soliton-like hyperplanes in spacetime whose quantum dynamics are governed by the theory of open strings whose ends are constrained to move on them (Fig. 1.5).

We will not attempt any description of the theory $\mathcal{U}$, which at present is not very well understood. Rather, we wish to focus on the remarkable
impact in high-energy physics that the discovery of D-branes has provided. Amongst other things, they have led to:

- Explicit realizations of nonperturbative string dualities [Polchinski (1995)]. For example, an elementary closed string state in Theory A (which is perturbative because its amplitudes are functions of the string coupling $g_s$) gets mapped under an S-duality transformation to a D-brane state in the dual Theory B (which depends on $1/g_s$ and is therefore nonperturbative).

- A microscopic explanation of black hole entropy and the rate of emission of thermal (Hawking) radiation for black holes in string theory [Callan and Maldacena (1996); Strominger and Vafa (1996)].

- The gauge theory/gravity (or AdS/CFT) correspondence [Aharony et al (2000); Maldacena (1998)]. D-branes carry gauge fields, while on the other hand they admit a dual description as solutions of the classical equations of motion of string theory and supergravity. Demanding that these two descriptions be equivalent implies, for
some special cases, that string theory is equivalent to a gauge field theory. This is an explicit realization of the old ideas that Yang–Mills theory may be represented as some sort of string theory.

- Probes of short-distances in spacetime [Douglas et al (1997)], where quantum gravitational fluctuations become important and classical general relativity breaks down.
- Large radius compactifications, whereby extra compact dimensions of size $\gg (\text{TeV})^{-1}$ occur [Antoniadis et al (1998); Arkani-Hamed, Dimopoulos and Dvali (1998)]. This is the distance scale probed in present-day accelerator experiments, which has led to the hope that the extra dimensions required by string theory may actually be observable.
- Brane world scenarios, in which we model our world as a D-brane [Randall and Sundrum (1999a); Randall and Sundrum (1999b)]. This may be used to explain why gravity couples so weakly to matter, i.e. why the effective Planck mass in our 3+1-dimensional world is so large, and hence gives a potential explanation of the hierarchy problem $m_P \gg m_{\text{weak}}$. 

Fig. 1.4. The space $\mathcal{U}$ of quantum string vacua. At each node a weakly-coupled string description is possible.
In what follows we will give the necessary background into the description of D-brane dynamics which leads to these exciting developments in theoretical high-energy physics.
Chapter 2

Classical String Theory

In this chapter we will start making the notions of the previous chapter quantitative. We will treat the classical dynamics of strings, before moving on in the next chapter to the quantum theory. Usually, at least at introductory levels, one is introduced to quantum field theory by using “second quantization” which is based on field operators that create or destroy quanta. Here we will describe string dynamics in “first quantization” instead, which is a sum-over-histories or Feynman path integral approach closely tied to perturbation theory. To familiarize ourselves with this geometrical description, we will start by explaining how it can be used to reproduce the well-known dynamics of a massive, relativistic point particle. This will easily introduce the technique that will readily generalize to the case of extended objects such as strings and D-branes. We will then describe the bosonic string within this approach and analyse its classical equations of motion. For the remainder of this book we will work in natural units whereby \( c = \hbar = G = 1 \).

2.1 The Relativistic Particle

Consider a particle which propagates in \( d \)-dimensional “target spacetime” \( \mathbb{R}^{1,d-1} \) with coordinates

\[
(t, \vec{x}) = x^\mu = (x^0, x^1, \ldots, x^{d-1}) . \quad (2.1)
\]

It sweeps out a path \( x^\mu(\tau) \) in spacetime, called a “worldline” of the particle, which is parametrized by a proper time coordinate \( \tau \in \mathbb{R} \) (Fig. 2.1). The infinitesimal, Lorentz-invariant path length swept out by the particle is

\[
\text{d}l = (-\text{d}s^2)^{1/2} = (-\eta_{\mu\nu} \, \text{d}x^\mu \, \text{d}x^\nu)^{1/2} \equiv (-\text{d}x^\mu \, \text{d}x_\mu)^{1/2} , \quad (2.2)
\]
where \( l \) is the proper-time of the particle and
\[
(\eta_{\mu\nu}) = \begin{pmatrix}
-1 & 0 \\
0 & 1 \\
\end{pmatrix}
\]
is the flat Minkowski metric. The action for a particle of mass \( m \) is then given by the total length of the trajectory swept out by the particle in spacetime:
\[
S[x] = -m \int dl(\tau) = -m \int d\tau \sqrt{-\dot{x}^\mu \dot{x}_\mu}
\]
where
\[
\dot{x}^\mu \equiv \frac{dx^\mu}{d\tau}.
\]
The minima of this action determine the trajectories of the particle with the smallest path length, and therefore the solutions to the classical equations of motion are the geodesics of the free particle in spacetime.

**Exercise 2.1.** Show that the Euler–Lagrange equations resulting from the action (2.4) give the usual equations of relativistic particle kinematics:
\[
p^\nu = 0 , \quad p'^\nu = \frac{m \dot{x}^\nu}{\sqrt{-\dot{x}^\rho \dot{x}_\rho}}.
\]

**2.1.1 Reparametrization Invariance**

The Einstein constraint \( p^2 \equiv p^\mu p_\mu = -m^2 \) on the classical trajectories of the particle (Exercise 2.1) is related to the fact that the action (2.4) is invariant under arbitrary, local reparametrizations of the worldline, i.e.
\[
\tau \longrightarrow \tau' , \quad \frac{d\tau}{d\tau'} > 0 .
\]
This is a kind of local worldline “gauge invariance” which means that the form of $S[x]$ is unchanged under such a coordinate change, since

$$d\tau \sqrt{-\dot{x}^\mu(\tau)\dot{x}_\mu(\tau)} = d\tau' \sqrt{-\dot{x}^{\mu}(\tau')\dot{x}_\mu(\tau')}.$$  (2.7)

It is a one-dimensional version of the usual four-dimensional general coordinate invariance in general relativity, in the sense that it corresponds to a worldline diffeomorphism symmetry of the theory. An application of the standard Noether procedure to the 0+1-dimensional field theory (2.4) leads to a conserved Noether current whose continuity equation is precisely the constraint $p^2 = -m^2$. It tells us how to eliminate one of the $p^\mu$'s, and in the quantum theory it becomes the requirement that physical states and observables must be gauge invariant. It enables us to select a suitable gauge. Let us look at a couple of simple examples, as this point will be crucial for our later generalizations.

### 2.1.2 Examples

**Example 2.1.** The static gauge choice corresponds to taking

$$x^0 = \tau \equiv t$$  (2.8)
in which case the action assumes the simple form
\[ S[x] = -m \int dt \sqrt{1 - \vec{v}^2} \]  
(2.9)

where
\[ \vec{v} = \frac{d\vec{x}}{dt} \]  
(2.10)
is the velocity of the particle. The equations of motion in this gauge take the standard form of those for a free, massive relativistic particle:
\[ \frac{d\vec{p}}{dt} = 0, \quad \vec{p} = m \vec{v} \sqrt{1 - \vec{v}^2}. \]  
(2.11)

**Example 2.2.** An even simpler gauge choice, known as the Galilean gauge, results from selecting
\[ \dot{x}^\mu \dot{x}_\mu = -1. \]  
(2.12)
The momentum of the particle in this gauge is given by the non-relativistic form (cf. Exercise 2.1)
\[ p^\mu = m \dot{x}^\mu, \]  
(2.13)
and the equations of motion are therefore
\[ \ddot{x}^\mu = 0 \]  
(2.14)
whose solutions are given by the Galilean trajectories
\[ x^\mu(\tau) = x^\mu(0) + p^\mu \tau. \]  
(2.15)

### 2.2 The Bosonic String

In the previous section we analysed a point particle, which is a zero-dimensional object described by a one-dimensional worldline in spacetime. We can easily generalize this construction to a *string*, which is a one-dimensional object described by a two-dimensional “worldsheet” that the string sweeps out as it moves in time with coordinates
\[ (\xi^0, \xi^1) = (\tau, \sigma). \]  
(2.16)
Here \(0 \leq \sigma \leq \pi\) is the spatial coordinate along the string, while \(\tau \in \mathbb{R}\) describes its propagation in time. The string’s evolution in time is described by functions \(x^\mu(\tau, \sigma), \mu = 0, 1, \ldots, d - 1\) giving the shape of its worldsheet.
in the target spacetime (Fig. 2.2). The “induced metric” \( h_{ab} \) on the string worldsheet corresponding to its embedding into spacetime is given by the “pullback” of the flat Minkowski metric \( \eta_{\mu\nu} \) to the surface,

\[
h_{ab} = \eta_{\mu\nu} \frac{\partial}{\partial \xi^a} x^\mu \frac{\partial}{\partial \xi^b} x^\nu ,
\]

(2.17)

where

\[
\partial_a \equiv \frac{\partial}{\partial \xi^a} , \quad a = 0, 1 .
\]

(2.18)

An elementary calculation shows that the invariant, infinitesimal area element on the worldsheet is given by

\[
dA = \sqrt{-\det_{a,b} (h_{ab})} \, d^2 \xi ,
\]

(2.19)

where the determinant is taken over the indices \( a, b = 0, 1 \) of the \( 2 \times 2 \) symmetric nondegenerate matrix \( (h_{ab}) \).

Fig. 2.2 The embedding \( (\tau, \sigma) \rightarrow x^\mu(\tau, \sigma) \) of a string trajectory into \( d \)-dimensional spacetime. As \( \tau \) increases the string sweeps out its two-dimensional worldsheet in the target space, with \( \sigma \) giving the position along the string.

In analogy to the point particle case, we can now write down an action whose variational law minimizes the total area of the string worldsheet.

---

1Notation: Greek letters \( \mu, \nu, \ldots \) denote spacetime indices, beginning Latin letters \( a, b, \ldots \) denote worldsheet indices, and later Latin letters \( i, j, \ldots \) label spatial directions in the target space. Unless otherwise stated, we also adhere to the standard Einstein summation convention for summing over repeated upper and lower indices.
in spacetime:

\[
S[x] = -T \int dA = -T \int d^2 \xi \sqrt{-\det_{a,b} (\partial_a x^\mu \partial_b x_\mu)} . \tag{2.20}
\]

The quantity \( T \) has dimensions of mass per unit length and is the *tension* of the string. It is related to the “intrinsic length” \( \ell_s \) of the string by

\[
T = \frac{1}{2\pi \alpha'} , \quad \alpha' = \ell_s^2 . \tag{2.21}
\]

The parameter \( \alpha' \) is called the “universal Regge slope”, because the string vibrational modes all lie on linear parallel Regge trajectories with slope \( \alpha' \). The action (2.20) defines a 1+1-dimensional field theory on the string worldsheet with bosonic fields \( x^\mu(\tau, \sigma) \).

**Exercise 2.2.** Show that the action (2.20) is reparametrization invariant, i.e. if \( \xi \mapsto \xi'(\xi') \), then it takes the same form when expressed in terms of the new worldsheet coordinates \( \xi' \).

Evaluating the determinant explicitly in (2.20) leads to the form

\[
S[x] = -T \int d\tau \ d\sigma \ \sqrt{\dot{x}^2 x'^2 - (\dot{x} \cdot x')^2 .} \tag{2.22}
\]

where

\[
\dot{x}^\mu = \frac{\partial x^\mu}{\partial \tau} , \quad x'^\mu = \frac{\partial x^\mu}{\partial \sigma} . \tag{2.23}
\]

This is the form that the original string action appeared in and is known as the “Nambu–Goto action” [Goto (1971); Nambu (1974)]. However, the square root structure of this action is somewhat awkward to work with. It can, however, be eliminated by the fundamental observation that the